

# Upper Bounds for $\alpha$ -Domination Parameters

A. Gagarin

Department of Mathematics and Statistics

Acadia University

Wolfville, Nova Scotia, B4P 2R6

Canada

A. Poghosyan and V.E. Zverovich

Faculty of Computing, Engineering and Mathematical Sciences

University of the West of England

Bristol, BS16 1QY

UK

## Abstract

In this paper, we provide a new upper bound for the  $\alpha$ -domination number. This result generalises the well-known Caro-Roditty bound for the domination number of a graph. The same probabilistic construction is used to generalise another well-known upper bound for the classical domination in graphs. We also prove similar upper bounds for the  $\alpha$ -rate domination number, which combines the concepts of  $\alpha$ -domination and  $k$ -tuple domination.

Keywords: *Graph; Domination;  $\alpha$ -Domination;  $\alpha$ -Rate Domination*

## 1 Introduction

Domination is one of the fundamental concepts in graph theory with various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs [1, 6, 8, 13]. Dominating sets in graphs are natural models for facility location problems in operational research. An important role is played by multiple domination, for example  $k$ -dominating sets can be used for balancing efficiency and fault tolerance [8].

We consider undirected simple finite graphs. If  $G$  is a graph of order  $n$ , then  $V(G) = \{v_1, v_2, \dots, v_n\}$  is the set of vertices of  $G$  and  $d_i$  denotes the degree of  $v_i$ . Let  $N(v)$  denote the neighbourhood of a vertex  $v$  in  $G$ , and  $N[v] = N(v) \cup \{v\}$  be the closed neighbourhood of  $v$ . A set  $X \subseteq V(G)$  is called a *dominating set* if every vertex not in  $X$  is adjacent to at least one vertex in  $X$ . The minimum cardinality of a dominating set of  $G$  is the *domination number*  $\gamma(G)$ . A set  $X$  is called a  *$k$ -dominating set* if every vertex not in  $X$  has at least  $k$  neighbors in  $X$ . The minimum cardinality of a  $k$ -dominating set of  $G$  is the  *$k$ -domination number*  $\gamma_k(G)$ . A set  $X$  is called a  *$k$ -tuple dominating set* of  $G$  if for every vertex  $v \in V(G)$ ,  $|N[v] \cap X| \geq k$ . The minimum cardinality of a  $k$ -tuple dominating set of  $G$  is the  *$k$ -tuple domination number*  $\gamma_{\times k}(G)$ . The  $k$ -tuple domination number is only defined for graphs with  $\delta \geq k - 1$ . A number of upper bounds for the multiple domination numbers can be found in [5, 10, 11, 12, 17].

Let  $\alpha$  be a real number satisfying  $0 < \alpha \leq 1$ . A set  $X \subseteq V(G)$  is called an  *$\alpha$ -dominating set* of  $G$  if for every vertex  $v \in V(G) - X$ ,  $|N(v) \cap X| \geq \alpha d_v$ , i.e.  $v$  is adjacent to at least  $\lceil \alpha d_v \rceil$  vertices of  $X$ . The minimum cardinality of an  $\alpha$ -dominating set of  $G$  is called the  *$\alpha$ -domination number*  $\gamma_\alpha(G)$ . The  $\alpha$ -domination was introduced by Dunbar et al. [9]. It is easy to see that

$\gamma(G) \leq \gamma_\alpha(G)$ , and  $\gamma_{\alpha_1}(G) \leq \gamma_{\alpha_2}(G)$  for  $\alpha_1 < \alpha_2$ . Also,  $\gamma(G) = \gamma_\alpha(G)$  if  $\alpha$  is sufficiently close to 0.

For an arbitrary graph  $G$  with  $n$  vertices and  $m$  edges, denote by  $\delta = \delta(G)$  and  $\Delta = \Delta(G)$  the minimum and maximum vertex degrees of  $G$ , respectively. The following results are proved in [9]:

$$\frac{\alpha\delta n}{\Delta + \alpha\delta} \leq \gamma_\alpha(G) \leq \frac{\Delta n}{\Delta + (1 - \alpha)\delta} \quad (1)$$

and

$$\frac{2\alpha m}{(1 + \alpha)\Delta} \leq \gamma_\alpha(G) \leq \frac{(2 - \alpha)\Delta n - (2 - 2\alpha)m}{(2 - \alpha)\Delta}. \quad (2)$$

Interesting results on  $\alpha$ -domination perfect graphs can be found in [7]. The problem of deciding whether  $\gamma_\alpha(G) \leq k$  for a positive integer  $k$  is known to be *NP*-complete [9]. Therefore, it is important to have good upper bounds for the  $\alpha$ -domination number and efficient approximation algorithms for finding ‘small’  $\alpha$ -dominating sets.

For  $0 < \alpha \leq 1$ , the  $\alpha$ -degree of a graph  $G$  is defined as follows:

$$\hat{d}_\alpha = \hat{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^n \left( \binom{d_i}{\lceil \alpha d_i \rceil} - 1 \right).$$

In this paper, we use a probabilistic approach to prove that

$$\gamma_\alpha(G) \leq \left( 1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1+1/\hat{\delta}} \hat{d}_\alpha^{1/\hat{\delta}}} \right) n,$$

where  $\hat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1$ . This result generalises the well-known upper bound of Caro and Roditty ([13], p. 48). Using the same probabilistic construction, we also show that

$$\gamma_\alpha(G) \leq \frac{\ln(\hat{\delta} + 1) + \ln \hat{d}_\alpha + 1}{\hat{\delta} + 1} n,$$

which generalises another well-known upper bound of Alon and Spencer [3], Arnaoutov [4], Lovász [15] and Payan [16]. Finally, we introduce the  $\alpha$ -rate domination number, which combines together the concepts of  $\alpha$ -domination and  $k$ -tuple domination, and show that the  $\alpha$ -rate domination number satisfies two similar upper bounds. The random constructions used in this paper also provide randomized algorithms to find  $\alpha$ -dominating and  $\alpha$ -rate dominating sets satisfying corresponding bounds.

## 2 New Upper Bounds for the $\alpha$ -Domination Number

One of the strongest known upper bounds for the domination number is due to Caro and Roditty:

**Theorem 1 (Caro and Roditty [13], p. 48)** *For any graph  $G$  with  $\delta \geq 1$ ,*

$$\gamma(G) \leq \left( 1 - \frac{\delta}{(1 + \delta)^{1+1/\delta}} \right) n. \quad (3)$$

The upper bound (3) is generalised for the  $\alpha$ -domination number in Theorem 2. Indeed, if  $d_i$  are fixed for all  $i = 1, \dots, n$ , and  $\alpha$  is sufficiently close to 0, then  $\hat{\delta} = \delta$  (provided  $\delta \geq 1$ ) and  $\hat{d}_\alpha = 1$ .

**Theorem 2** For any graph  $G$ ,

$$\gamma_\alpha(G) \leq \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \widehat{d}_\alpha^{1/\widehat{\delta}}}\right) n, \quad (4)$$

where  $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1$ .

**Proof:** Let  $A$  be a set formed by an independent choice of vertices of  $G$ , where each vertex is selected with the probability

$$p = 1 - \left(\frac{1}{(1 + \widehat{\delta}) \widehat{d}_\alpha}\right)^{1/\widehat{\delta}}. \quad (5)$$

Let us denote

$$B = \{v_i \in V(G) - A : |N(v_i) \cap A| \leq \lceil \alpha d_i \rceil - 1\}.$$

It is obvious that the set  $D = A \cup B$  is an  $\alpha$ -dominating set. The expectation of  $|D|$  is

$$\begin{aligned} E(|D|) &= E(|A|) + E(|B|) \\ &= \sum_{i=1}^n P(v_i \in A) + \sum_{i=1}^n P(v_i \in B) \\ &= pn + \sum_{i=1}^n \sum_{r=0}^{\lceil \alpha d_i \rceil - 1} \binom{d_i}{r} p^r (1-p)^{d_i-r-1}. \end{aligned}$$

It is easy to see that, for  $0 \leq r \leq \lceil \alpha d_i \rceil - 1$ ,

$$\binom{d_i}{r} \leq \binom{d_i}{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{r}.$$

Also,

$$d_i - \lceil \alpha d_i \rceil \geq \lfloor \delta(1 - \alpha) \rfloor.$$

Therefore,

$$\begin{aligned} E(|D|) &\leq pn + \sum_{i=1}^n \binom{d_i}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \sum_{r=0}^{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{r} p^r (1-p)^{\lceil \alpha d_i \rceil - 1 - r} \\ &= pn + \sum_{i=1}^n \binom{d_i}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \\ &\leq pn + (1-p)^{\lfloor \delta(1-\alpha) \rfloor + 2} \widehat{d}_\alpha n \\ &= pn + (1-p)^{\widehat{\delta} + 1} \widehat{d}_\alpha n \\ &= \left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \widehat{d}_\alpha^{1/\widehat{\delta}}}\right) n. \end{aligned} \quad (6)$$

Note that the value of  $p$  in (5) is chosen to minimize the expression (6). Since the expectation is an average value, there exists a particular  $\alpha$ -dominating set of order at most  $\left(1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \widehat{d}_\alpha^{1/\widehat{\delta}}}\right) n$ , as required. The proof of the theorem is complete.  $\blacksquare$

Notice that in some cases Theorem 2 provides a much better bound than the upper bound in (1). For example, if  $G$  is a 1000-regular graph, then Theorem 2 gives  $\gamma_{0.1}(G) < 0.305n$ , while (1) yields only  $\gamma_{0.1}(G) < 0.527n$ .

**Corollary 1** For any graph  $G$ ,

$$\gamma_\alpha(G) \leq \frac{\ln(\widehat{\delta} + 1) + \ln \widehat{d}_\alpha + 1}{\widehat{\delta} + 1} n. \quad (7)$$

**Proof:** We put

$$p = \min \left\{ 1, \frac{\ln(\widehat{\delta} + 1) + \ln \widehat{d}_\alpha}{\widehat{\delta} + 1} \right\}.$$

Using the inequality  $1 - p \leq e^{-p}$ , we can estimate the expression (6) as follows:

$$E(|D|) \leq pn + e^{-p(\widehat{\delta}+1)} \widehat{d}_\alpha n.$$

If  $p = 1$ , then the result easily follows. If  $p = \frac{\ln(\widehat{\delta}+1)+\ln \widehat{d}_\alpha}{\widehat{\delta}+1}$ , then

$$E(|D|) \leq \frac{\ln(\widehat{\delta} + 1) + \ln \widehat{d}_\alpha + 1}{\widehat{\delta} + 1} n,$$

as required. ■

Corollary 1 generalises the following well-known upper bound independently proved by several authors [3, 4, 15, 16]:

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n. \quad (8)$$

### 3 $\alpha$ -Rate Domination

Define a set  $X \subseteq V(G)$  to be an  $\alpha$ -rate dominating set of  $G$  if for any vertex  $v \in V(G)$ ,

$$|N[v] \cap X| \geq \alpha d_v.$$

Let us call the minimum cardinality of an  $\alpha$ -rate dominating set of  $G$  the  $\alpha$ -rate domination number  $\gamma_{\times\alpha}(G)$ . It is easy to see that  $\gamma_\alpha(G) \leq \gamma_{\times\alpha}(G)$ . The concept of  $\alpha$ -rate domination is similar to the well-known  $k$ -tuple domination (for example, see [14, 17]). For  $0 < \alpha \leq 1$ , the closed  $\alpha$ -degree of a graph  $G$  is defined as follows:

$$\widetilde{d}_\alpha = \widetilde{d}_\alpha(G) = \frac{1}{n} \sum_{i=1}^n \left( \binom{d_i + 1}{\lceil \alpha d_i \rceil} - 1 \right).$$

In fact, the only difference between the  $\alpha$ -degree and the closed  $\alpha$ -degree is that to compute the latter we choose from  $d_i + 1$  vertices instead of  $d_i$ , i.e. from the closed neighborhood  $N[v_i]$  of  $v_i$  instead of  $N(v_i)$ .

The following theorem provides an analogue of the Caro-Roditty bound (Theorem 1) for the  $\alpha$ -rate domination number:

**Theorem 3** For any graph  $G$  and  $0 < \alpha \leq 1$ ,

$$\gamma_{\times\alpha}(G) \leq \left( 1 - \frac{\widehat{\delta}}{(1 + \widehat{\delta})^{1+1/\widehat{\delta}} \widetilde{d}_\alpha^{1/\widehat{\delta}}} \right) n, \quad (9)$$

where  $\widehat{\delta} = \lfloor \delta(1 - \alpha) \rfloor + 1$ .

**Proof:** Let  $A$  be a set formed by an independent choice of vertices of  $G$ , where each vertex is selected with probability  $p$ ,  $0 \leq p \leq 1$ . For  $m \geq 0$ , denote by  $B_m$  the set of vertices  $v \in V(G)$  dominated by exactly  $m$  vertices of  $A$  and such that  $|N[v] \cap A| < \alpha d_v$ , i.e.

$$|N[v] \cap A| = m \leq \lceil \alpha d_v \rceil - 1.$$

Note that each vertex  $v \in V(G)$  is in at most one of the sets  $B_m$  and  $0 \leq m \leq \lceil \alpha d_v \rceil - 1$ . We form a set  $B$  in the following way: for each vertex  $v \in B_m$ , select  $\lceil \alpha d_v \rceil - m$  vertices from  $N(v)$  that are not in  $A$  and add them to  $B$ . Consider the set  $D = A \cup B$ . It is easy to see that  $D$  is an  $\alpha$ -rate dominating set. The expectation of  $|D|$  is:

$$\begin{aligned} E(|D|) &\leq E(|A|) + E(|B|) \\ &\leq \sum_{i=1}^n P(v_i \in A) + \sum_{i=1}^n \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} (\lceil \alpha d_i \rceil - m) P(v_i \in B_m) \\ &= pn + \sum_{i=1}^n \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} (\lceil \alpha d_i \rceil - m) \binom{d_i + 1}{m} p^m (1-p)^{d_i + 1 - m} \\ &\leq pn + \sum_{i=1}^n \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{m} p^m (1-p)^{d_i + 1 - m} \\ &= pn + \sum_{i=1}^n \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \sum_{m=0}^{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{m} p^m (1-p)^{\lceil \alpha d_i \rceil - 1 - m} \\ &= pn + \sum_{i=1}^n \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} (1-p)^{d_i - \lceil \alpha d_i \rceil + 2} \\ &\leq pn + (1-p)^{\lfloor \delta(1-\alpha) \rfloor + 2} \sum_{i=1}^n \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} \\ &= pn + (1-p)^{\hat{\delta} + 1} \tilde{d}_\alpha n, \end{aligned}$$

since

$$(\lceil \alpha d_i \rceil - m) \binom{d_i + 1}{m} \leq \binom{d_i + 1}{\lceil \alpha d_i \rceil - 1} \binom{\lceil \alpha d_i \rceil - 1}{m}.$$

Thus,

$$E(|D|) \leq pn + (1-p)^{\hat{\delta} + 1} \tilde{d}_\alpha n. \quad (10)$$

Minimizing the expression (10) with respect to  $p$ , we obtain

$$E(|D|) \leq \left( 1 - \frac{\hat{\delta}}{(1 + \hat{\delta})^{1 + 1/\hat{\delta}} \tilde{d}_\alpha^{1/\hat{\delta}}} \right) n,$$

as required. The proof of Theorem 3 is complete. ■

**Corollary 2** For any graph  $G$ ,

$$\gamma_{\times \alpha}(G) \leq \frac{\ln(\hat{\delta} + 1) + \ln \tilde{d}_\alpha + 1}{\hat{\delta} + 1} n. \quad (11)$$

**Proof:** Using an approach similar to that in the proof of Corollary 1, the result follows if we put

$$p = \min \left\{ 1, \frac{\ln(\widehat{\delta} + 1) + \ln \widetilde{d}_\alpha}{\widehat{\delta} + 1} \right\}$$

and use the inequality  $1 - p \leq e^{-p}$  to estimate the expression (10) as follows:

$$E(|D|) \leq pn + e^{-p(\widehat{\delta}+1)} \widetilde{d}_\alpha n.$$

■

Note that, similar to Corollary 1, the bound of Corollary 2 also generalises the classical upper bound (8). However, the probabilistic construction used to obtain the bounds (9) and (11) is different from that to obtain the bounds (4) and (7).

## 4 Final Remarks and Open Problems

Notice that the concept of the  $\alpha$ -rate domination number  $\gamma_{\times\alpha}(G)$  is ‘opposite’ to the  $\alpha$ -independent  $\alpha$ -domination number  $i_\alpha(G)$  as defined in [7]. It would be interesting to use a probabilistic method construction to obtain an upper bound for  $i_\alpha(G)$ .

Also, the random constructions used to obtain the upper bounds (4), (7), (9) and (11) provide randomized algorithms to find corresponding dominating sets in a given graph  $G$ . It would be interesting to derandomize these algorithms or to obtain independent deterministic algorithms to find corresponding dominating sets satisfying the upper bounds (4), (7), (9) and (11). Algorithms approximating the  $\alpha$ - and  $\alpha$ -rate domination numbers up to a certain degree of precision would be interesting too. For the  $k$ -tuple domination number, an interesting approximation algorithm was found by Klasing and Laforest [14].

Using probabilistic methods, Alon [2] proved that the bound (8) is asymptotically best possible. More precisely, it was proved that when  $n$  is large there exists a graph  $G$  such that

$$\gamma(G) \geq \frac{\ln(\delta + 1) + 1}{\delta + 1} n(1 + o(1)).$$

We wonder if a similar result can be proved for the bounds (7) and (11), and conjecture that when  $n$  is large enough there exist graphs  $G$  and  $H$  such that

$$\gamma_\alpha(G) \geq \frac{\ln(\widehat{\delta} + 1) + \ln \widehat{d}_\alpha + 1}{\widehat{\delta} + 1} n(1 + o(1))$$

and

$$\gamma_{\times\alpha}(H) \geq \frac{\ln(\widehat{\delta} + 1) + \ln \widetilde{d}_\alpha + 1}{\widehat{\delta} + 1} n(1 + o(1)).$$

## References

- [1] J. Alber, N. Betzler and R. Niedermeier, Experiments on data reduction for optimal domination in networks. *Annals of Operations Research*, **146** (2006) 105–117.
- [2] N. Alon, Transversal numbers of uniform hypergraphs. *Graphs and Combin.* **6** (1990) 1–4.
- [3] N. Alon and J.H. Spencer, *The Probabilistic Method*, John Wiley & Sons, Inc., New York, 1992.

- [4] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices. *Prikl. Mat. i Programirovanie* **11** (1974) 3–8 (in Russian).
- [5] Y. Caro and R. Yuster, Dominating a family of graphs with small connected subgraphs. *Combinatorics, Probability and Computing*, **9** (2000) 309–313.
- [6] C. Cooper, R. Klasing and M. Zito, Lower bounds and algorithms for dominating sets in web graphs. *Internet Math.* **2** (2005) 275–300.
- [7] F. Dahme, D. Rautenbach and L. Volkmann,  $\alpha$ -Domination perfect trees. *Discrete Math.* (2007), doi:10.1016/j.disc.2007.06.043, in press.
- [8] F. Dai and J. Wu, On Constructing  $k$ -Connected  $k$ -Dominating Set in Wireless Ad Hoc and Sensor Networks. *J. Parallel and Distributed Computing*, **66** (7)(2006) 947–958.
- [9] J.E. Dunbar, D.G. Hoffman, R.C. Laskar and L.R. Markus,  $\alpha$ -Domination. *Discrete Math.* **211** (2000) 11–26.
- [10] O. Favaron, On a conjecture of Fink and Jacobson concerning  $k$ -domination and  $k$ -dependence. *J. Combin Theory Ser. B*, **39** (1985) 101–102.
- [11] O. Favaron, A. Hansberg and L. Volkmann, On  $k$ -domination and minimum degree in graphs. *J. Graph Theory* **57** (2008) 33–40.
- [12] A. Gagarin and V.E. Zverovich, A generalized upper bound for the  $k$ -tuple domination number. *Discrete Math.* **308** (5-6)(2008) 880–885, doi:10.1016/j.disc.2007.07.033.
- [13] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
- [14] R. Klasing and C. Laforest, Hardness results and approximation algorithms of  $k$ -tuple domination in graphs. *Inform. Process. Letters* **89** (2004) 75–83.
- [15] L. Lovász, On the ratio of optimal integral and fractional covers. *Discrete Math.* **13** (1975) 383–390.
- [16] C. Payan, Sur le nombre d’absorption d’un graphe simple. *Cahiers Centre Études Recherche Opér.* **17** (1975), no. 2-4, 307–317 (in French).
- [17] V.E. Zverovich, The  $k$ -tuple domination number revisited. *Applied Math. Letters* (2007), doi:10.1016/j.aml.2007.10.016, in press.